UNRAVELLING THE MYSTERY OF RANK REVERSAL IN THE AHP

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1. *Type #1*: the **final rank order** of the alternatives changes if an **irrelevant alternative** is added to (or removed from) the problem. See, for example, Buede & Maxwell (1995), Zanakis et al. (1998), Wang & Luo (2009), García-Cascales & Lamata (2012), Verly & De Smet (2013) and Cinelli et al. (2014).

2. *Type #2*: the indication of **the best alternative** changes if a **non-optimal alternative** is replaced by another worse one. See, for example, Triantaphyllou (2001) and Wang & Triantaphyllou (2008).

3. *Type #3*: the **transitivity** property is violated if an **irrelevant alternative** is added to (or removed from) the problem. See, for example, Triantaphyllou (2001) and Wang & Triantaphyllou (2008).

4. *Type #4*: the **transitivity** property is violated if the initial decision-problem is decomposed into **sub-problems**, i. e., for the same decision problem and when the same MCDM method is used, the rankings of the smaller problems are in conflict with the overall ranking of the alternatives. See, for example, Triantaphyllou (2001) and Wang & Triantaphyllou (2008).

5. *Type #5*: the **final rank order** of the alternatives changes if a **non-discriminating criterion** is removed from the problem. See, for example, Finan & Hurley (2002), Lin et al. (2008), Jung et al. (2009) and Jan et al. (2011) and Verly & De Smet (2013).

The phenomenon of rank reversal was illustrated by Belton and Gear

Table 5.12 Pairwise comparison matrix of the criteria

	a	*b*	*c*
a	1	1	1
b	1	1	1
c	1	1	1

Table 5.13 Pairwise comparison matrix of the alternatives with regard to the criterion *a*

	A	*B*	*C*
A	1	1/9	1
B	9	1	9
C	1	1/9	1

Table 5.14 Pairwise comparison matrix of the alternatives with regard to the criterion *b*

	A	*B*	*C*
A	1	9	9
B	1/9	1	1
C	1/9	1	1

Table 5.15 Pairwise comparison matrix of the alternatives with regard to the criterion *c*

	A	*B*	*C*
A	1	8/9	8
B	9/8	1	9
C	1/8	1/9	1

Table 5.16 Score matrix

	a	b	c
A	0.091	0.818	0.444
B	0.818	0.091	0.500
C	0.091	0.091	0.056

the priority vector of criteria

$$w = [1/3\ 1/3\ 1/3]^T$$

Finally, the global alternative priorities are calculated

$$v = Sw = [0.451\ 0.470\ 0.079]$$

The final ranking of the alternatives (from best to worst) is B–A–C.

Table 5.17 New pairwise comparison matrix of the alternatives with regard to the criterion *a*

	A	*B*	*C*	*D*
A	1	1/9	1	1/9
B	9	1	9	1
C	1	1/9	1	1/9
D	9	1	9	1

Table 5.18 New pairwise comparison matrix of the alternatives with regard to the criterion *b*

	A	*B*	*C*	*D*
A	1	9	9	9
B	1/9	1	1	1
C	1/9	1	1	1
D	1/9	1	1	1

Table 5.19 New pairwise comparison matrix of the alternatives with regard to the criterion *c*

	A	*B*	*C*	*D*
A	1	8/9	8	8/9
B	9/8	1	9	1
C	1/8	1/9	1	1/9
D	9/8	1	9	1

Table 5.20 Score matrix

	a	b	c
A	0.050	0.750	0.300
B	0.450	0.083	0.333
C	0.050	0.083	0.037
D	0.450	0.083	0.333

the priority vector of criteria

$$w = [1/3 \ 1/3 \ 1/3]^T$$

Finally, the global alternative priorities are calculated

$$v = Sw = [0.365 \ 0.289 \ 0.057 \ 0.289]$$

The final ranking of the alternatives

(from best to worst) is A–B–D–C

$$v = Sw = [0.451\ 0.470\ 0.079]$$

The final ranking of the alternatives (from best to worst) is B–A–C.

$$v = Sw = [0.365\ 0.289\ 0.057\ 0.289]$$

The final ranking of the alternatives (from best to worst) is A–B–D–C. We observe that the ranking has changed. In the initial example, B was preferred over A, but now A is preferred over B. There was no change in the relative preferences of A over B between the two examples, so the fact that the overall preference does not remain unchanged causes the rank reversal phenomenon.

Considered is the true priority vector w

w=[7/20, 1/4, 1/4, 3/20] w=[0.35, 0.25, 0.25, 0.15]

and $\boldsymbol{A}(w)$ derived from that w presented as follows:

$$\begin{bmatrix} 1 & 7/5 & 7/5 & 7/3 \\ 5/7 & 1 & 1 & 5/3 \\ 5/7 & 1 & 1 & 5/3 \\ 3/7 & 3/5 & 3/5 & 1 \end{bmatrix}$$

Then, $\boldsymbol{R}(x)$ is considered which is produced by a hypothetical DM. It is assumed that DM is very trustworthy and is able to express judgments very precisely at the same time being still somehow limited by the necessity of expressing judgments on a scale (the example utilizes Saaty's scale). In this scenario, entries of $\boldsymbol{A}(w)$ are rounded to Saaty's scale and the entries are made reciprocal – the principal condition for Pairwise Comparison Matrices (PCM) applied in the AHP:

It should be noted that $\boldsymbol{R}(x)$ is perfectly consistent.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}$$

Table 1 – Values of CI_{REV} and CI_{GM} as well proposed quality characteristics of w estimates – w_E derived from $\boldsymbol{R}(x)$ with application of the REV and GM method

PM(*)	ESTIMATES – w_E	PERFORMANCE MEASURES		
		CI_{PM}	AAE	SRC
REV	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966
GM	$[0.285714, 0.285714, 0.285714, 0.142857]^T$	0.0	0.0357143	0.8164966

(*) PM stands for prioritization method

the true priority vector w

$$AAE = \frac{1}{n}\sum_{i=1}^{n}|w_i - w_{Ei}|$$

w=[7/20, 1/4, 1/4, 3/20]

w=[0.35, 0.25, 0.25, 0.15]

$$v = Sw = [0.451 \; 0.470 \; 0.079]$$

The final ranking of the alternatives (from best to worst)

B-A-C cannot be determined indisputably !!!

For n=3 → MEDIAN **AAE_i** = **0.0138389** x **2** =

0.0276778 > 0.019 (0.470-0.451) **!!**

MEAN **AAE_i** = **0.0537636** x **2** = **0.1075272 !!!**

For n=3 → MEDIAN **$MaxDEV_i$** FROM **AAE_i** = **0.0108732**

MEAN **$MaxDEV_i$** FROM **AAE_i** = **0.0299839**

THUS: for n=3 → **MAX ABS DEV** for a **PRIORITY RATIO (PR)**:

MEDIAN for **MAX DEV(PR)** = **0.0247121** (0.0138389+0.0108732)

→ 0.0247121 x 2 = **0.054242 !!**

MEAN for **MAX DEV(PR)** = **0.0837475** (0.0537636+0.0299839)

→ 0.0837475 x 2 = **0.167495 !!**

Conclusions

The evidence of the examination indicates that Priority Vectors derived from both consistent and inconsistent Pairwise Comparison Matrices are fuzzy and should not be considered as set but only as estimated with certain level of probability. Hence, any evidence showing rank reversal in the AHP models which is based on assumptions about their determined value should be considered as erroneous.